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Construction of exact Riemannian instanton solutions

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Abstract
We present the exact construction of Riemannian (or stringy) instantons, which are classical solutions of 2D Yang–Mills theories that interpolate between initial and final string configurations. They satisfy the Hitchin equations with special boundary conditions. For the case of $U(2)$ gauge group those equations can be written as the sinh-Gordon equation with a delta-function source. Using the techniques of integrable theories based on the zero curvature conditions, we show that the solution is a condensate of an infinite number of one-solitons with the same topological charge and with all possible rapidities.

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1. Introduction

In this paper we intend to prove the existence of *Riemannian* or *stringy instantons*. The name is due to the fact that these are classical solutions of a 2D $U(N)$ YM theory that interpolate between initial and final string configurations. In other words they describe Riemann surfaces with punctures, where the latter represent asymptotic entering and exiting strings. In the simplest case ($N = 2$), proving the existence of Riemannian instantons amounts to finding exact solutions of the sinh-Gordon equation with a delta-function source (see below for more details). To be definite let $a = a(z)$ be a polynomial in the complex variable z, with distinct roots, and let us introduce a new variable ζ defined by $\frac{\delta\zeta}{\delta w} = \sqrt{a}$, where $z = \mathrm{e}^{w}$, w being the coordinate on an infinite cylinder. The equation we want to solve is

$$\partial_\zeta \partial_{\bar\zeta} u - 2g^2 \sinh 2u = -\frac{\pi}{4}\delta(a)(\partial_\zeta a)(\partial_{\bar\zeta}\bar a) \tag{1.1}$$

which has to be understood in the sense of complex distribution theory. In this equation g is a constant coupling. An equivalent way to state (1.1) is to write the usual sinh-Gordon equation

$$\partial_\zeta \partial_{\bar\zeta} u - 2g^2 \sinh 2u = 0 \tag{1.2}$$

and to look for solutions which, near the zeros of a, behave like

$$u \sim -\tfrac{1}{2}\ln|a| \qquad \text{for} \quad a \sim 0. \tag{1.3}$$

We will also require that u vanishes as $z \to 0$ and $z \to \infty$. This kind of equation was met for the first time in the framework of matrix string theory in [1–4], and solutions to these equations, satisfying (1.3), were shown to exist only numerically. In particular, in [4] this was done in the framework of a square lattice approximation, assuming a very simple form for a. The same type of solutions appeared in the context of form factors and correlation functions for the Ising model in [5]. Radial reductions of the sinh-Gordon equation (which boil down to Painlevé III equations) were considered in [6, 7].

This paper is devoted to proving the existence of solutions to the above equations, with the desired boundary conditions, in an analytic way, and to give their closed expressions in terms of the modified Bessel function K_0. In fact, the validity of the solution relies on some nonlinear differential identities satisfied by integrals of K_0, which to our knowledge, have not yet appeared in the literature.

The central ideas of the proof is (1) to use the Leznov–Saveliev approach and (2) to view the solution as a condensate of solitons.

In more detail, we begin by writing the sinh-Gordon equation (1.1) in terms of zero curvature conditions, which include besides the usual Lax–Zakharov–Shabat equation a second relation leading to non-local conservation laws. Such generalized zero curvature conditions follow from the ideas proposed in [8] to study integrable theories in any dimension. Once we have the equations of motion written in terms of a zero curvature condition, we utilize the Leznov–Saveliev method to construct the corresponding Riemannian instanton solution. This method uses the fact that the dynamical variables of the system are contained in the zero curvature potentials: the sinh-Gordon φ field appears as a parameter of the group element we use in order to write the flat connection A_μ as

$$A_\mu = -\partial_\mu W W^{-1} = \text{function of a group element } \gamma \qquad \gamma = \mathrm{e}^{\varphi H^0 + \nu C}. \tag{1.4}$$

Thus, due to the path independence encoded in $F_{\mu\nu} = 0$, we are able to write the group element W in distinct forms. This, compounded with some properties of the Kac–Mooody algebra, leads us to a simple algebraic relation for the 'group parameters' φ and ν, i.e.

$$\langle\lambda|\gamma^{-1}|\lambda\rangle = \langle\lambda|\gamma_+ N_+ M_-^{-1}\gamma_-|\lambda\rangle \tag{1.5}$$

where the elements $\gamma_\pm$, N_+ and M_- have nice properties in terms of Kac–Moody algebra representations. To determine the two parameters φ and ν, we make use of two highest weight representations. This provides us with a relation for φ in terms of the expected values of the group elements N_+ and M_-. So, solving the sinh-Gordon equation is equivalent to furnish these two elements. Once this is done, we conveniently choose the parameters in such a way that the boundary conditions (1.3) are satisfied.

The key point in this construction is that in order to obtain the desired solution we must choose the constant group elements of the solitonic specialization of the Leznov–Saveliev construction as an infinite product of exponentials of vertex operators. The product is in fact a continuous one, since it involves all possible values of the rapidities of the one-solitons. In addition, all one-solitons entering the expansion have the same topological charge. This leads us to interpret such a configuration as a condensate of solitons. As was realized in [9], this type of solution can be written as a Fredholm determinant, and our solution is similar to the one found in [5], where correlation functions of the Ising model were obtained in terms of $\tau^{(N)}$ functions of the sinh-Gordon model, with $N \to \infty$. In order to arrive at the true solution a continuum limit must be taken for the condensate of solitons and the Fredholm determinant must be rewritten as an infinite series of integrals, whose convergence conditions are studied and particular normalizations are fixed in order to satisfy the required boundary conditions. The solution heuristically derived in this way is finally shown to satisfy equation (1.1) or (1.2) plus (1.3).

Before we enter into the very existence proof it is worth reviewing the framework where equation (1.1) arises and plays a fundamental role. Matrix string theory (MST) [10–12] is the theory that arises upon compactifying the matrix theory [13] on a circle [14]. It is expected to be a non-perturbative version of type IIA string theory. An attempt to substantiate such a conjecture was started in [1, 2] and completed in [3, 4, 15], where it was shown that a correspondence between MST and type IIA theory exists not only at the tree level, but that actually MST contains the full perturbative expansion of type IIA string theory. It was in this context that (1.1) appeared.

Looking for classical solutions of MST that preserve half supersymmetry, the following system of equations was found:

$$\begin{aligned} &F_{w\bar{w}} - \mathrm{i}g^2[X, \bar{X}] = 0 \\ &D_w\bar{X} = 0 \qquad D_{\bar{w}}X = 0 \end{aligned} \tag{1.6}$$

where $F_{w\bar{w}}$ denotes the curvature of a connection with components A_w and $A_{\bar{w}}$, while X is an $N \times N$ matrix and $\bar{X}$ its Hermitian conjugate. $D_w, D_{\bar{w}}$ denote the covariant derivatives with respect to $A_w, A_{\bar{w}}$. Equations (1.6) may be called *Hitchin equations*, because they were discussed first by Hitchin in a different context [16], or *Riemannian instanton equations* because of their geometrical interpretation. To elucidate this terminology and the importance of these equations let us consider the simplest case, in which the gauge group is $U(2)$. The problem to be solved is finding a couple (A, X) that satisfies (1.6). To this end we choose the following ansatz:

$$X = Y^{-1}MY \qquad A_w = \mathrm{i}\partial_w Y^\dagger (Y^{-1})^\dagger \tag{1.7}$$

where Y is a suitable matrix $\in SL(2, \mathbb{C})$, and M is the following 2×2 matrix:

$$M = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix} \tag{1.8}$$

where a is a function on the complex plane. As a consequence of the equation $D_{\bar{w}}X = 0$, it follows that $\partial_{\bar{w}}a = 0$, i.e. a is holomorphic in z (at least for finite z). As explained above, we will assume that a is a polynomial in z with distinct roots. Now, given such an a we want to find Y so that (1.6) is satisfied. We parametrize Y as $Y = \begin{pmatrix} \mathrm{e}^p & 0 \\ 0 & \mathrm{e}^{-p} \end{pmatrix}$ where $p = \frac{u}{2} + \frac{1}{4}\ln|a|$, and u is a function to be determined. Then using (1.7) we find

$$X = \begin{pmatrix} 0 & a\mathrm{e}^{-2p} \\ \mathrm{e}^{2p} & 0 \end{pmatrix} \qquad A_w = \mathrm{i}\partial_w p \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.9}$$

Now, it is easy to verify that the first equation in (1.6) implies

$$2\partial_w\partial_{\bar{w}}p - g^2(\mathrm{e}^{4p} - |a|^2\,\mathrm{e}^{-4p}) = 0. \tag{1.10}$$

Inserting the explicit form of p and the change of variable $w \to \zeta$, s.t. $\frac{\delta\zeta}{\delta w} = \sqrt{a}$, one can rewrite (1.10) as (1.1). If u is a smooth solution to this equation, the couple (X, A) is a solution to (1.6) which is smooth everywhere except perhaps at infinity. Now, the important point is that the matrix M represents a branched covering of the z-plane. This is seen by diagonalizing M by means of a matrix in $SL(2, \mathbb{C})$:

$$M = S\widehat{M}S^{-1} \qquad \widehat{M} = \begin{pmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{pmatrix} \qquad S = \frac{\mathrm{i}}{\sqrt{2}}\begin{pmatrix} a^{\frac{1}{4}} & a^{\frac{1}{4}} \\ a^{-\frac{1}{4}} & -a^{-\frac{1}{4}} \end{pmatrix}. \tag{1.11}$$

The two eigenvalues of M represent the two branches of the equation

$$y^2 = a \tag{1.12}$$

which is the defining equation of a hyperelliptic Riemann surface, with branch points corresponding to the roots of a. Therefore, the solution at issue represents a Riemann surface, which justifies the adjective in the name *Riemannian instanton*. The instanton nature of this solution is due to the fact that (1.6) is the two-dimensional reduction of the YM self-duality equation in 4D. In [17] this example was analysed in great detail. It was shown there that, if the u solution to (1.1) satisfies the boundary conditions stated at the beginning of the introduction, the matrix X, outside the branch points of a and when $g \to \infty$, becomes $\widehat{M}$ up to a unitary transformation. This fact plays a crucial role in establishing the correspondence between MST and type IIA theory, see [3, 4].

It is, therefore, of upmost importance to establish the existence of the above solutions of (1.6) and, therefore, of the corresponding u solutions of (1.1). On the other hand, it is clear that showing the existence of exact solutions of (1.1) is a problem interesting in itself.

The paper is organized as follows. In section 2 we apply the Leznov–Saveliev method to our problem and define the soliton condensate. In section 3 we verify that what has been heuristically constructed in section 2 is in fact the looked for solution. An appendix is devoted to clarifying a few technical problems encountered in the course of the proof.

2. Construction of solution through the Leznov–Saveliev algebraic method

In this section we review the Leznov–Saveliev method [18] for the construction of solutions of affine Toda-type theories, based on the zero curvature formulation of two-dimensional integrable systems. Even though we work in two dimensions we stick to the point of view of higher dimensional integrable models, which can be constructed from two potentials, A and B [8]. Among other things, such an approach leads to the construction of new conserved currents, not obtained via the usual two-dimensional formalism [8].

2.1. Zero curvature condition

Equation (1.1) admits a representation in terms of the following zero curvature conditions:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0 \tag{2.1}$$

$$D^\mu B_\mu = \partial^\mu B_\mu + [A^\mu, B_\mu] = 0. \tag{2.2}$$

In two dimensions the condition (2.1) is the well-known Lax–Zakharov–Shabat equation. In dimensions higher than 2, equations (2.1) and (2.2) were shown to be sufficient local conditions for the vanishing of the generalized zero curvature equations relevant for higher dimensional integrable theories [8]. Here we apply equations (2.1) and (2.2) for the two-dimensional model (1.1), in particular the relation (2.2) leads to non-local conservation laws. The procedure applies equally well to a wide class of two-dimensional integrable models, such as the Abelian and non-Abelian Toda models (affine or not), possessing a representation in terms of Lax–Zakharov–Shabat equation (2.1). That equation can be enriched by the extra conservation laws (2.2) without any further restriction in their dynamics, and we plan to analyse that in more detail in a future publication.

Let $\hat{\mathcal{G}}$ be an affine $sl(2)$ Kac–Moody algebra. We take the local zero curvature potentials as

$$\begin{aligned} A_w &\equiv -\partial_w \gamma\gamma^{-1} + E_{-1} \qquad & A_{\bar{w}} &\equiv \gamma E_1 \gamma^{-1} \\ B_w &\equiv P^\psi(E_{-1}) \qquad & B_{\bar{w}} &\equiv P^\psi(\gamma E_1 \gamma^{-1}) \end{aligned} \tag{2.3}$$

where

$$\gamma \equiv e^{\varphi H^0 + \nu C} \tag{2.4}$$

and

$$E_1 \equiv g T_+^0 + g\bar{a}(\bar{z}) T_-^1 \qquad E_{-1} \equiv g a(z) T_+^{-1} + g T_-^0. \tag{2.5}$$

Following [8] we have taken A_μ and B_μ to belong to a non-semisimple Lie algebra formed by the $sl(2)$ Kac–Moody algebra and an Abelian ideal which transforms under its adjoint representation P^ψ, i.e.

$$\begin{aligned}
&\left[T_a^m, T_b^n\right] = f_{ab}^c T_c^{m+n} + \mathrm{Tr}(T_a T_b) C \delta_{m+n,0} \\
&\left[T_a^m, P^\psi\left(T_b^n\right)\right] = P^\psi\left(\left[T_a^m, T_b^n\right]\right) \\
&\left[P^\psi\left(T_a^m\right), P^\psi\left(T_b^n\right)\right] = 0.
\end{aligned} \tag{2.6}$$

With this choice, the zero curvature conditions (2.1) lead us to the system of differential equations

$$\partial_{\bar{w}} \partial_w \varphi = g^2 (e^{2\varphi} - |a|^2 e^{-2\varphi}) \tag{2.7}$$

$$\partial_{\bar{w}} \partial_w \nu = g^2 |a|^2 e^{-2\varphi}. \tag{2.8}$$

The identification

$$\varphi = 2p \equiv u + \tfrac{1}{2} \ln |a| \tag{2.9}$$

turns (2.7) into the equation

$$2\partial_w \partial_{\bar{w}} p = g^2 (e^{4p} - |a|^2 e^{-4p}). \tag{2.10}$$

This is exactly the sinh-Gordon equation (1.1) once we make the change of variables $w \to \zeta$, such that (see appendix A)

$$\frac{d\zeta}{dw} = \sqrt{a} \qquad \frac{d\bar{\zeta}}{d\bar{w}} = \sqrt{\bar{a}}. \tag{2.11}$$

As we pointed out in the introduction, the sinh-Gordon equation with source is equivalent to the homogeneous equation together with the following boundary conditions:

$$u \sim -\tfrac{1}{2} \ln |a| \qquad \varphi \sim \text{finite} \qquad \text{for} \quad a \sim 0. \tag{2.12}$$

So, u must diverge logarithmically at the zeros of a, and from (2.9) it follows that φ should be finite there. On the other hand, far away from any zero of a (i.e. at $z = 0, \infty$) we need

$$u \sim 0 \qquad \varphi \sim \tfrac{1}{2} \ln |a| \qquad \text{for} \quad a \sim \infty. \tag{2.13}$$

One can check that the condition (2.2) is trivially satisfied by the potentials (2.3), i.e. it holds true for any field configuration including those which are not solutions of the equations of motion (2.7) and (2.8). However, due to (2.1) it follows that the connection A_μ is flat, and so there is a group element W such that

$$A_\mu \equiv -\partial_\mu W W^{-1}. \tag{2.14}$$

Consequently, it follows from (2.2) that the currents

$$J_\mu \equiv W^{-1} B_\mu W \tag{2.15}$$

are conserved

$$\partial^\mu J_\mu = 0. \tag{2.16}$$

The group element W only exists for field configurations that satisfy the equations of motion, and it is non-local in the field variables. Consequently, the currents (2.15) are non-local. We intend to study the properties of these currents, in a wide class of models, in a future publication.

2.2. Leznov–Saveliev construction

The Leznov–Saveliev method uses some features of the Kac–Moody algebra representations, as well as some geometrical properties due to the zero curvature condition (2.1). Using this machinery, and as a consequence of (2.3), we will be able to write the general solution of the model as

$$\mathrm{e}^{-\varphi} = \frac{\langle\lambda_1|N_+(\bar{w})M_-^{-1}(w)|\lambda_1\rangle}{\langle\lambda_0|N_+(\bar{w})M_-^{-1}(w)|\lambda_0\rangle}\,\mathrm{e}^{\theta_+-\theta_-} \tag{2.17}$$

$$\mathrm{e}^{-\nu} = \langle\lambda_0|N_+(\bar{w})M_-^{-1}(w)|\lambda_0\rangle\,\mathrm{e}^{\xi_+-\xi_-}. \tag{2.18}$$

Here, besides the fields φ and ν, we introduced the auxiliary fields ξ_+, ξ_-. θ_+, θ_- appear as parameters that will be fixed by the boundary condition. N_+, M_- are group elements which obey the equations

$$\partial_{\bar{w}}N_+N_+^{-1} = -g\left(\mathrm{e}^{-2\theta_+}T_+^0 + \bar{a}(\bar{w})\mathrm{e}^{2\theta_+}T_-^1\right) \tag{2.19}$$

$$\partial_w M_-M_-^{-1} = -g\left(a(w)\mathrm{e}^{-2\theta_-}T_+^{-1} + \mathrm{e}^{2\theta_-}T_-^0\right). \tag{2.20}$$

The general solution to (2.7), and so of (1.1), is given by (2.17). Therefore, the Riemannian instanton should correspond to some particular choice of the parameters. This means that we have a solution to the model once we specify the parameters $\theta_\pm, \xi_\pm$, and solve (2.19) and (2.20) for M_-, N_+.

Note that the group element W introduced in (2.14) has to be regular, since otherwise A_μ will not be flat. Remember that in order for A_μ to satisfy the zero curvature condition it is necessary that the derivatives commute when acting on W. It then follows that the group elements g_1, g_2 and γ also have to be regular. By regular we mean a quantity such that derivatives ∂_w and $\partial_{\bar{w}}$ commute when acting on it. From (see [21])

$$\partial_{\bar{z}}z^{-k-1} = (-1)^k\frac{\pi}{k!}\delta^{(k,0)}(\bar{z}, z) \tag{2.21}$$

one observes that

$$\partial_z\partial_{\bar{z}}z^{-k-1} = \partial_{\bar{z}}\partial_z z^{-k-1}. \tag{2.22}$$

Therefore, the fields φ and ν, as well as the parameters $\theta_\pm$ and $\xi_\pm$, can have log singularities, so that the corresponding group elements will have at most poles.

2.3. The Riemannian instanton solutions

As we have seen, the general solution of the model (2.7), or equivalently (1.1), is given by (2.17). We now have to choose the parameters and integration constants of the general solution in such a way as to obtain the Riemannian instanton solutions with the properties described in the introduction. We begin by choosing the functions $\theta_\pm$ as

$$\theta_+ = -\tfrac{1}{4}\ln\bar{a} \qquad \theta_- = \tfrac{1}{4}\ln a. \tag{2.23}$$

This choice simplifies the integration of the elements N_+ and M_-. Indeed, (2.19) and (2.20) become

$$\partial_{\bar{w}}N_+N_+^{-1} = -g\sqrt{\bar{a}(\bar{w})}\;b_1 \qquad \partial_w M_-M_-^{-1} = -g\sqrt{a(w)}\;b_{-1}. \tag{2.24}$$

The operators b_1 and b_{-1} are elements of a Heisenberg subalgebra of the $\widehat{sl}(2)$ Kac–Moody algebra [19, 20]. The latter is an algebra of harmonic oscillators generated by

$$b_{2n+1} \equiv T_+^n + T_-^{n+1} \qquad [b_{2m+1}, b_{2n+1}] = C(2m+1)\delta_{m+n+1,0}. \tag{2.25}$$

We can then integrate (2.24)

$$N_+ = \mathrm{e}^{I_+ b_1} h_+ \qquad I_+ = -g \int \mathrm{d}\bar{w}\, \sqrt{\bar{a}(\bar{w})} = -g\bar{\zeta} \tag{2.26}$$

$$M_- = \mathrm{e}^{I_- b_{-1}} h_- \qquad I_- = -g \int \mathrm{d}w\, \sqrt{a(w)} = -g\zeta \tag{2.27}$$

where we used the change of variables to (2.11), and where $h_\pm$ are constant group elements obtained by exponentiating the affine Kac–Moody algebra (integration constants).

We now return to the general solution (2.17). With our choice of $\theta_\pm$ (2.23) and in view of (2.9) we get

$$\mathrm{e}^{-u} = \frac{\langle \lambda_1 | N_+ M_-^{-1} | \lambda_1 \rangle}{\langle \lambda_0 | N_+ M_-^{-1} | \lambda_0 \rangle}. \tag{2.28}$$

Here we come to a crucial point in the construction of the Riemannian instanton solution. As is well known [22, 23], the one-soliton solutions are obtained by taking the integration constants $h_\pm$, such that $h_+ h_-^{-1} = \mathrm{e}^{V(\mu)}$, where $V(\mu)$ is an element of the Kac–Moody algebra which is an eigenstate of the oscillators b_{2n+1}, i.e.

$$[b_{2n+1}, V(\mu)] = -2\mu^{2n+1} V(\mu). \tag{2.29}$$

The operator $V(\mu)$ is expressed in terms of a special basis of the Kac–Moody algebra. The nice properties of such an operator are best appreciated in the principal vertex operator representation of the Kac–Moody algebra. An important relation satisfied by $V(\mu)$ is given by

$$V(z)V(w) = :: V(z)V(w) :: \left(\frac{z-w}{z+w}\right)^2. \tag{2.30}$$

From it we observe that

$$V(\mu)V(\nu) \to 0 \qquad \text{for} \quad \mu \to \nu. \tag{2.31}$$

This implies that the exponential $\mathrm{e}^{V(\mu)}$ is truncated at first order, and so we do not have convergence problems in our expressions. Such a property is what makes the vertex operator representation deserve the name of integral representation [19, 20]. It also explains the truncation of Hirota's expansion of the tau functions, since those are nothing more than special expectation values of $V(\mu)$ in the states of the vertex operator representation [23].

If, for instance, one takes $a(z)$ to be constant and chooses the integration constants $h_\pm$, such that $h_+ h_-^{-1} = \mathrm{e}^{V(\mu)}$, then one obtains from (2.28) the one-soliton solution to the sinh-Gordon equation (by taking $u \to \mathrm{i}u$ one gets the sine-Gordon one-soliton). The parameter μ is related to the rapidity θ of the soliton through $\mu \equiv \epsilon \mathrm{e}^{\theta}$, with $\epsilon = \pm 1$. It is ϵ that determines the sign of the topological charge (in the case of sine-Gordon) and makes the difference between the soliton and anti-soliton solutions.

The n-soliton solution is obtained by taking $h_+ h_-^{-1}$ as a product of those exponentials, i.e. $h_+ h_-^{-1} = \prod_{i=1}^{n} \mathrm{e}^{V(\mu_i)}$. As we now explain, the Riemannian instanton solution is obtained by taking $h_+ h_-^{-1}$ to be a *continuous* infinite product of exponentials $\mathrm{e}^{V(\mu_i)}$. In fact, we shall take the product in such a way that exponentials for smaller values of μ_i appear on the left, and we vary μ_i continuously from zero to $+\infty$. In addition, each value of μ_i appears only once, without repetition. So, what we have is an N-soliton solution, with $N \to \infty$, where all the rapidities appear once, and we do not have a mixture of soliton and anti-solitons since the μ_i are all positive. Therefore, we have some sort of soliton condensate[5].

[5] We are indebted to Olivier Babelon for pointing out to us that the Riemannian instanton solution should have such structure. His intuition came from his experience with D Bernard on the calculations of form factors and correlation functions for the Ising model [5].

In order to build up the solution we start with an infinite *discrete* product of exponentials, and later take the continuous limit. So, we take the constant h_+, h_- in (2.26), (2.27) to be

$$h_+ h_-^{-1} \equiv \prod_{i=1}^{\infty} \mathrm{e}^{V(\mu_i)}. \tag{2.32}$$

From this point on we explore some useful properties of the vertex operators and their action on the representation states, which will allow us to end up with a closed expression for the solution u. So, once the constants h_+, h_- have been chosen as in (2.32), the solution (2.28) depends on

$$\langle\lambda|N_+ M_-^{-1}|\lambda\rangle = \langle\lambda|\prod_{i=1}^{\infty}(1 + \mathrm{e}^{\beta(\mu_i)} V(\mu_i))|\lambda\rangle \qquad \beta(\mu_i) \equiv 2g\left(\mu_i\bar{\zeta} + \frac{\zeta}{\mu_i}\right) \tag{2.33}$$

where we have used (2.27), (2.29) and (2.31). We can expand (2.33) in terms of sums as

$$\begin{aligned}
\langle\lambda_0|N_+ M_-^{-1}|\lambda_0\rangle &= 1 + \sum_i \mathrm{e}^{\beta(\mu_i)} + \sum_{i<j} \mathrm{e}^{\beta(\mu_i)}\,\mathrm{e}^{\beta(\mu_j)} \left(\frac{\mu_j - \mu_i}{\mu_j + \mu_i}\right)^2 \\
&\quad + \sum_{i<j<k} \mathrm{e}^{\beta(\mu_i)}\,\mathrm{e}^{\beta(\mu_j)}\,\mathrm{e}^{\beta(\mu_k)} \left(\frac{\mu_j - \mu_i}{\mu_j + \mu_i}\right)^2 \left(\frac{\mu_k - \mu_i}{\mu_k + \mu_i}\right)^2 \left(\frac{\mu_k - \mu_j}{\mu_k + \mu_j}\right)^2 + \cdots \\
\langle\lambda_1|N_+ M_-^{-1}|\lambda_1\rangle &= 1 - \sum_i \mathrm{e}^{\beta(\mu_i)} + \sum_{i<j} \mathrm{e}^{\beta(\mu_i)}\,\mathrm{e}^{\beta(\mu_j)} \left(\frac{\mu_j - \mu_i}{\mu_j + \mu_i}\right)^2 \\
&\quad - \sum_{i<j<k} \mathrm{e}^{\beta(\mu_i)}\,\mathrm{e}^{\beta(\mu_j)}\,\mathrm{e}^{\beta(\mu_k)} \left(\frac{\mu_j - \mu_i}{\mu_j + \mu_i}\right)^2 \left(\frac{\mu_k - \mu_i}{\mu_k + \mu_i}\right)^2 \left(\frac{\mu_k - \mu_j}{\mu_k + \mu_j}\right)^2 + \cdots.
\end{aligned} \tag{2.34}$$

Remember that we take the μ_i to be real and positive, and that two μ_i never coincide.

Expressions (2.34) can be written in the form of Fredholm determinants. A similar result was found in [5], where exact correlation functions of the Ising model were shown to be related to the tau functions of the sinh-Gordon model. Following [5] we get

$$\begin{aligned}
\langle\lambda_0|N_+ M_-^{-1}|\lambda_0\rangle &= \det(1 + W) \\
\langle\lambda_1|N_+ M_-^{-1}|\lambda_1\rangle &= \det(1 - W)
\end{aligned} \tag{2.35}$$

where W is the matrix

$$W_{ij} \equiv \mathrm{e}^{\beta(\mu_i)/2} \frac{\sqrt{4\mu_i\mu_j}}{\mu_i + \mu_j} \mathrm{e}^{\beta(\mu_j)/2}. \tag{2.36}$$

Using (2.28) one then gets

$$u = \ln\left(\frac{\det(1 + W)}{\det(1 - W)}\right) = \mathrm{Tr}\ln\frac{1 + W}{1 - W} \tag{2.37}$$

where we used $\ln\det M = \mathrm{Tr}\ln M$. Expanding the logarithm one gets

$$u = 2\sum_{n=0}^{\infty} \frac{\mathrm{Tr}\, W^{2n+1}}{2n + 1}. \tag{2.38}$$

2.4. The continuous limit

As we have said, we want to take the limit where the infinite product of exponentials in (2.32) becomes a continuous one. In order to do that we take the label i of the parameter μ_i to be the rapidity θ of the soliton, and let it run from $-\infty$ to ∞. We then have that $\mu_i \to \mu_\theta = \mathrm{e}^{\theta}$. Then

$$\sum_i \to \Lambda \int_{-\infty}^{\infty} \mathrm{d}\theta = \Lambda \int_0^{\infty} \frac{\mathrm{d}\mu}{\mu} \tag{2.39}$$

where Λ is a scaling factor of the integration measure.

Then, from (2.36) it follows that[6]

$$\mathrm{Tr}\, W^N \to \Lambda^N \int_0^{\infty} \frac{\mathrm{d}\mu_1}{\mu_1} \cdots \int_0^{\infty} \frac{\mathrm{d}\mu_N}{\mu_N} \mathrm{e}^{\beta(\mu_1)/2} \frac{\sqrt{4\mu_1\mu_2}}{\mu_1+\mu_2} \mathrm{e}^{\beta(\mu_2)} \frac{\sqrt{4\mu_2\mu_3}}{\mu_2+\mu_3} \mathrm{e}^{\beta(\mu_3)} \cdots \\ \times \mathrm{e}^{\beta(\mu_N)} \frac{\sqrt{4\mu_N\mu_1}}{\mu_N+\mu_1} \mathrm{e}^{\beta(\mu_1)/2}. \tag{2.40}$$

Therefore, (2.38) becomes

$$u = 2\sum_{n=0}^{\infty} \frac{(2\Lambda)^{2n+1}}{2n+1} I_{2n+1} \tag{2.41}$$

with

$$I_N \equiv \int_0^{\infty} \frac{\mathrm{d}\mu_1}{\mu_1} \cdots \int_0^{\infty} \frac{\mathrm{d}\mu_N}{\mu_N} \frac{\mu_1}{(\mu_1+\mu_2)} \frac{\mu_2}{(\mu_2+\mu_3)} \cdots \frac{\mu_N}{(\mu_N+\mu_1)} \mathrm{e}^{\beta(\mu_1)+\cdots+\beta(\mu_N)}. \tag{2.42}$$

In the case $N = 1$, we have that

$$I_1 = \frac{1}{2}\int_0^{\infty} \frac{\mathrm{d}\mu}{\mu} \mathrm{e}^{\beta(\mu)} = K_0(4|g||\zeta|) \tag{2.43}$$

where K_0 is the modified Bessel function. In fact the above expression is valid for $\mathrm{Re}(g\zeta) < 0$. However, we shall take it to be valid for $\mathrm{Re}(g\zeta) > 0$ too (see comments below (2.49)).

Note that the integrals I_N are real. Indeed, from (2.33) we have

$$\beta^*(\mu_i) = \beta\left(\frac{1}{\mu_i}\right). \tag{2.44}$$

Therefore, one can undo the complex conjugation with the change of integration variables, $\mu_i \to 1/\mu_i$, since $\int_0^{\infty} \frac{\mathrm{d}\mu_i}{\mu_i}$ is left unchanged. In addition, $\mu_i/(\mu_i+\mu_j) \to \mu_j/(\mu_i+\mu_j)$, and so the product of those terms in the integrand of (2.42) is left invariant. Consequently, the solution u given in (2.41) is real (since Λ is real).

We now want to analyse the boundary conditions satisfied by the solution (2.41). In order to do that we perform the change of integration variables

$$\phi_i \equiv \ln \frac{\mu_i}{\mu_{i+1}} \qquad i = 1, 2, \ldots, N-1 \qquad \nu \equiv \left(\prod_{i=1}^{N} \mu_i\right)^{1/N}. \tag{2.45}$$

The integrals (2.42) become

$$I_N = \frac{1}{2^N}\int_{-\infty}^{\infty} \mathrm{d}\phi_1 \cdots \mathrm{d}\phi_{N-1} \frac{1}{\cosh\left(\frac{1}{2}\phi_1\right)\cosh\left(\frac{1}{2}\phi_2\right)\cdots\cosh\left(\frac{1}{2}\phi_{N-1}\right)\cosh\left(\frac{1}{2}\sum_{n=1}^{N-1}\phi_n\right)} \\ \times \int_0^{\infty} \frac{\mathrm{d}\nu}{\nu} \mathrm{e}^{2g\left(\bar{\zeta} f_N(\phi)\nu + \zeta f_N(-\phi)\frac{1}{\nu}\right)} \tag{2.46}$$

[6] Note that the subindices of μ have a different meaning now. They label the variables giving the values of rows and columns of the matrices, and not the actual values of those as before.

where

$$f_N(\phi) \equiv \sum_{l=1}^{N} \exp\left(\frac{1}{N}\left(-\sum_{n=1}^{l-1} n\phi_n + \sum_{n=l}^{N-1}(N-n)\phi_n\right)\right). \tag{2.47}$$

If $\mathrm{Re}(g\zeta) < 0$, the integral in ν in (2.46) is the modified Bessel function K_0, and so one gets

$$I_N = \frac{1}{2^{N-1}} \int_{-\infty}^{\infty} \mathrm{d}\phi_1 \cdots \mathrm{d}\phi_{N-1} \frac{K_0(4|g\|\zeta|\sqrt{w_N})}{\cosh\left(\frac{1}{2}\phi_1\right)\cosh\left(\frac{1}{2}\phi_2\right)\cdots\cosh\left(\frac{1}{2}\phi_{N-1}\right)\cosh\left(\frac{1}{2}\sum_{n=1}^{N-1}\phi_n\right)} \tag{2.48}$$

where w_N is given by

$$w_N \equiv f_N(\phi) f_N(-\phi) = N + 2\sum_{l=0}^{N-2}\sum_{j=1}^{N-l-1} \cosh \sum_{i=j}^{j+l} \phi_i. \tag{2.49}$$

However, we shall take the expression (2.48) to also be valid for $\mathrm{Re}(g\zeta) > 0$. Such an analytical continuation process will be justified later when we shall check the validity of the solution by directly substituting it into the equations of motion. Therefore, the solution (2.41) depends on ζ and g through their norms only.

2.5. The boundary conditions

For large arguments the modified Bessel function K_0 has the following behaviour:

$$K_0(y) \sim \sqrt{\frac{\pi}{2y}}\,\mathrm{e}^{-y}\left(1 + O\left(\frac{1}{y}\right)\right) \qquad \text{for large } y. \tag{2.50}$$

Consequently, it is clear that

$$I_N \to 0 \qquad \text{for} \quad |\zeta| \to \infty \tag{2.51}$$

and so, the u field does go to zero for large ζ, as required (see (2.13)). Indeed, near $z = 0, \infty$ the variable ζ diverges, see appendix A.

The analysis for small ζ is trickier. The reason is that taking $|\zeta|$ small does not guarantee that the argument of the Bessel function K_0 is small, since w_N can be infinitely large. However, in the region where $|\zeta|w_N$ diverges for small $|\zeta|$ the function K_0 vanishes and so there is no contribution to the integral I_N. Therefore, we can use the following reasoning: let $|\zeta|$ have a fixed infinitesimal value $|\zeta| = \varepsilon$. We split the domain of integration into two regions, namely,

$$\begin{aligned} D_0 &\equiv \text{region of } (\phi_1, \ldots, \phi_{N-1}) \qquad \text{where} \quad \varepsilon 4|g|\sqrt{w_N} < \sqrt{\varepsilon} \\ D_1 &\equiv \text{region of } (\phi_1, \ldots, \phi_{N-1}) \qquad \text{where} \quad \varepsilon 4|g|\sqrt{w_N} > \sqrt{\varepsilon}. \end{aligned} \tag{2.52}$$

In the region D_0 we use the fact that for small arguments, K_0 diverges as

$$K_0(y) \sim -\ln\frac{z}{2}(1 + O(y^2)) \qquad \text{for small } y \tag{2.53}$$

and so

$$\begin{aligned} I_N = {} & \frac{1}{2^{N-1}} \int_{D_0} \mathrm{d}\phi_1 \cdots \mathrm{d}\phi_{N-1} \frac{-\ln(2|g||\zeta|\sqrt{w_N})}{\cosh\left(\frac{1}{2}\phi_1\right)\cosh\left(\frac{1}{2}\phi_2\right)\cdots\cosh\left(\frac{1}{2}\phi_{N-1}\right)\cosh\left(\frac{1}{2}\sum_{n=1}^{N-1}\phi_n\right)} \\ & + \frac{1}{2^{N-1}} \int_{D_1} \mathrm{d}\phi_1 \cdots \mathrm{d}\phi_{N-1} \\ & \times \frac{K_0(4|g||\zeta|\sqrt{w_N})}{\cosh\left(\frac{1}{2}\phi_1\right)\cosh\left(\frac{1}{2}\phi_2\right)\cdots\cosh\left(\frac{1}{2}\phi_{N-1}\right)\cosh\left(\frac{1}{2}\sum_{n=1}^{N-1}\phi_n\right)}. \end{aligned} \tag{2.54}$$

Note that in the region D_1 the argument of K_0 never vanishes and so K_0 is finite there. On the other hand, on D_1 we have

$$w_N > \frac{1}{4|g|\sqrt{\varepsilon}} \tag{2.55}$$

and so $w_N \to \infty$ as $\varepsilon \to 0$. But from (2.49) one observes that the only way for that to happen is that at least one of the ϕ_i should diverge. Therefore, the denominator of the integrand of (2.54), in the D_1 region, diverges. So, one gets that the integral in D_1 in (2.54) vanishes for $\varepsilon \to 0$.

Consequently, the integral in D_0 in (2.54) implies that

$$I_N \sim -\kappa_N \ln|\zeta| \qquad \text{for} \quad |\zeta| \to 0 \tag{2.56}$$

with

$$\kappa_N \equiv \frac{1}{2^{N-1}} \int_{-\infty}^{\infty} \frac{\mathrm{d}\phi_1 \cdots \mathrm{d}\phi_{N-1}}{\cosh\left(\frac{1}{2}\phi_1\right)\cosh\left(\frac{1}{2}\phi_2\right)\cdots\cosh\left(\frac{1}{2}\phi_{N-1}\right)\cosh\left(\frac{1}{2}\sum_{n=1}^{N-1}\phi_n\right)}. \tag{2.57}$$

Performing the integration using the fact that

$$\begin{aligned}\kappa_N &= \int_{-\infty}^{\infty} \mathrm{d}\phi_1 \cdots \mathrm{d}\phi_N \frac{\delta\left(\sum_{i=1}^N \phi_i\right)}{\prod_{i=1}^N \cosh\phi_i} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k \int_{-\infty}^{\infty} \mathrm{d}\phi_1 \cdots \mathrm{d}\phi_N \frac{\mathrm{e}^{\mathrm{i}k\sum_{i=1}^N \phi_i}}{\prod_{i=1}^N \cosh\phi_i} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k \left(\int_{-\infty}^{\infty} \mathrm{d}\phi \frac{\mathrm{e}^{\mathrm{i}k\phi}}{\cosh\phi}\right)^N \end{aligned} \tag{2.58}$$

one gets

$$\kappa_{2n+1} = \frac{\pi^{2n}}{2^n}\frac{(2n-1)!!}{n!}. \tag{2.59}$$

Consequently, we have from (2.41) that

$$u \sim -\frac{2}{\pi} f(2\pi\Lambda)\ln|\zeta| \qquad \text{for} \quad |\zeta| \to 0 \tag{2.60}$$

where

$$f(x) \equiv \sum_{n=0}^{\infty} \frac{(2n-1)!!}{2^n n!(2n+1)} x^{2n+1}. \tag{2.61}$$

Note that this series is convergent for $x^2 < 1$, and divergent for $x^2 > 1$. For $x^2 = 1$ the ratio test does not say anything, but one can check that it does converge there and $f(1) = \pi/2$. Therefore, we must have $|\Lambda| \leqslant 1/2\pi$.

Our function $a(z)$ is supposed to be a polynomial and to represent a hyperelliptic Riemann surface. As discussed in appendix A, a good local coordinate near a branch point z_i is $\xi_i = \sqrt{z - z_i}$, i.e. near z_i we have $z = z_i + \xi_i{}^2$. So, according to (A.6), near the branch point, we must have $\zeta \sim a^{3/2}$. Therefore,

$$u \sim -\frac{3}{\pi} f(2\pi\Lambda)\ln|a|. \tag{2.62}$$

One can check that

$$f\left(\frac{1}{2}\right) = \frac{\pi}{6}. \tag{2.63}$$

Consequently, in order to satisfy the boundary condition (2.12), we must set

$$\Lambda = \frac{1}{4\pi}. \tag{2.64}$$

Therefore, from (2.41), the desired solution to (1.1) is given by

$$u = 2\sum_{n=0}^{\infty}\frac{I_{2n+1}}{(2n+1)(2\pi)^{2n+1}} \tag{2.65}$$

where I_{2n+1} is given in (2.43) and (2.48).

We remark that (2.65) depends only on the combination $4|g||\zeta|$, and, as a consequence, it is symmetric in ζ under rotation around the origin $\zeta = 0$. This symmetry came as a bonus of our solution, it was not an initial input. It is worth stressing this point to mark the difference with other papers [6, 7], where this symmetry was assumed from the very beginning. Another important difference of our solution is that it turns out to be symmetric in the ζ variable, not in the original w or z coordinates. ζ is a rather complicated variable and, as a coordinate, it needs a few specifications, which will be provided in appendix A.

3. Check of the solution

As we have seen in (2.64), the value of the scaling factor Λ of the integration measure, introduced in (2.39), was fixed to $1/4\pi$. This was imposed by the behaviour of the solution at $\zeta = 0$ (or equivalently at the zeros of $a(z)$). However, as we will see in this section, the solution holds true for any value of Λ, outside the zeros of $a(z)$. That is a consequence of very interesting nonlinear differential equations satisfied by the integrals I_{2n+1} defined in (2.48). Therefore, in order to emphasize those properties, we shall not fix Λ in this check of the solution.

We begin by looking at the convergence of the series (2.41). As we have seen in (2.51), the integrals I_{2n+1} go to zero for large arguments. Therefore, we should not have problems of convergence of the series (2.41) for large arguments. For ζ close to zero one can use (2.56) and (2.59) and the ratio test to check that the series converges for $\Lambda < \sqrt{2}/2\pi$. Therefore, the series (2.65) for the final solution should converge everywhere.

Let us come now to the actual check. This will be done by using the variable ζ. As explained in appendix A this is a good coordinate for any generic value, except in correspondence with the branch points, where ζ vanishes, and near $z = 0$, $z = \infty$ where ζ diverges. These points, therefore, should be treated separately. Now, as far as the region near $|\zeta| = \infty$ is concerned, the question is simple. In fact we have already noted that K_0 exponentially vanishes there, see (2.50), so that all I_N and their derivatives vanish too, (2.51). Therefore, equation (1.2) is satisfied in the limit $\zeta \to \infty$.

Let us consider now the verification for finite values of ζ. We begin by calculating the derivatives of the field u close to $\zeta = 0$. From (2.60) it follows that

$$\partial_\zeta u \sim -\frac{1}{\pi} f(2\pi\Lambda)\frac{1}{\zeta} \tag{3.66}$$

and so using (2.21)

$$\partial_{\bar\zeta}\partial_\zeta u \sim -f(2\pi\Lambda)\delta(\zeta,\bar\zeta). \tag{3.67}$$

Some care must be taken here since we have been working with delta functions in different coordinate frames. In order to avoid misunderstandings which can lead to inconsistencies in fixing Λ, we devote a discussion to this point in appendix A. From (A.13) we see that

$$\partial_{\bar\zeta}\partial_\zeta u \sim -f(2\pi\Lambda)\delta(\zeta,\bar\zeta) = -f(2\pi\Lambda)\tfrac{3}{2}\delta(a,\bar a)\partial_\zeta a\partial_{\bar\zeta}\bar a \tag{3.68}$$

and using (2.63) and (2.64) we get

$$\partial_{\bar\zeta}\partial_\zeta u \sim -\frac{\pi}{6}\delta(\zeta,\bar\zeta) = -\frac{\pi}{4}\delta(a,\bar a)\,\partial_\zeta a\partial_{\bar\zeta}\bar a. \tag{3.69}$$

Therefore, we do find that (1.1) is satisfied at $\zeta = 0$.

Let us now evaluate the derivatives of u for $\zeta \neq 0$. Since I_{2n+1} depends on ζ through the modified Bessel function K_0, we consider

$$\partial_{\bar\zeta}\partial_\zeta K_0(4|g||\zeta|\sqrt{w_N}) = (4|g|)^2 w_N \partial_{\bar\zeta}|\zeta|\partial_\zeta|\zeta|K_0''(4|g||\zeta|\sqrt{w_N}) + 4|g|\sqrt{w_N}\partial_{\bar\zeta}\partial_\zeta|\zeta|K_0'(4|g||\zeta|\sqrt{w_N}). \tag{3.70}$$

Observe that

$$\partial_\zeta|\zeta| = \frac{1}{2}\sqrt{\frac{\bar\zeta}{\zeta}} \qquad \partial_{\bar\zeta}|\zeta| = \frac{1}{2}\sqrt{\frac{\zeta}{\bar\zeta}} \tag{3.71}$$

and with the help of (2.21) we obtain

$$\partial_{\bar\zeta}\partial_\zeta|\zeta| = \frac{1}{4|\zeta|} + \frac{\pi}{4}|\zeta|\delta(\zeta,\bar\zeta). \tag{3.72}$$

Since we are taking the point $\zeta = 0$ out, we can use the defining equation of K_0, namely,

$$z^2K_0''(z) + zK_0'(z) - z^2K_0(z) = 0 \tag{3.73}$$

and (3.71) and (3.72) to get

$$\partial_{\bar\zeta}\partial_\zeta K_0(4|g||\zeta|\sqrt{w_N}) = 4g^2 w_N K_0(4|g||\zeta|\sqrt{w_N}). \tag{3.74}$$

Therefore, from (2.48) one has

$$\partial_{\bar\zeta}\partial_\zeta I_N = 4g^2 J_N \qquad \zeta \neq 0 \tag{3.75}$$

where

$$J_N \equiv \frac{1}{2^{N-1}}\int_{-\infty}^{\infty} \mathrm{d}\phi_1\cdots\mathrm{d}\phi_{N-1}\frac{w_N K_0(4|g||\zeta|\sqrt{w_N})}{\cosh\left(\frac{1}{2}\phi_1\right)\cosh\left(\frac{1}{2}\phi_2\right)\cdots\cosh\left(\frac{1}{2}\phi_{N-1}\right)\cosh\left(\frac{1}{2}\sum_{n=1}^{N-1}\phi_n\right)}. \tag{3.76}$$

Note that the relation (3.75) is also valid for $N = 1$, with

$$J_1 = K_0(4|g||\zeta|). \tag{3.77}$$

The reason is that from (2.43) we have that $I_1 = K_0(4|g||\zeta|)$, and so (3.74) becomes (3.75) using the fact that $w_1 = 1$.

Using arguments similar to those leading to (3.74), one can check that I_N and J_N satisfy

$$z^2I_N''(z) + zI_N'(z) = z^2J_N(z) \tag{3.78}$$

where z stands for the argument of those functions, i.e. $z \equiv 4|g||\zeta|$.

We now have, from (2.41), (3.75) and (3.68), that

$$\partial_{\bar\zeta}\partial_\zeta u = 8g^2\sum_{n=0}^{\infty}\frac{(2\Lambda)^{2n+1}}{2n+1}J_{2n+1} - \delta(\zeta,\bar\zeta)f(2\pi\Lambda) \tag{3.79}$$

where $f(x)$ was defined in (2.61).

So, substituting into equation (1.1), we get

$$2g^2\left(4\sum_{n=0}^{\infty}\frac{(2\Lambda)^{2n+1}}{2n+1}J_{2n+1} - \sinh 2u\right) = \delta(\zeta,\bar\zeta)f(2\pi\Lambda) - \frac{\pi}{4}\delta(a)\partial_\zeta a\partial_{\bar\zeta}\bar a. \tag{3.80}$$

As we have already seen, the vanishing of the rhs of (3.80) fixes the value of Λ. Indeed, using (A.13) we see that we need to choose Λ such that $f(2\pi\Lambda) = \pi/6$. But this is exactly what we have done in (2.63) and (2.64) to get the right boundary conditions.

The lhs of (3.80), on the other hand, vanishes for any Λ. This is an amazing result and involves special properties of the Bessel function K_0, or more precisely of I_N and J_N. Expanding the lhs of (3.80) in powers of Λ we get

$$\begin{aligned}
&\Lambda \to K_0(4|g||\zeta|) = K_0(4|g||\zeta|) \qquad (I_1 = J_1)\\
&\Lambda^3 \to J_3 = I_3 + 8I_1^3\\
&\Lambda^5 \to J_5 = I_5 + \tfrac{40}{3}I_1^2 I_3 + \tfrac{32}{3}I_1^5\\
&\Lambda^7 \to J_7 = I_7 + \tfrac{224}{9}I_1^4 I_3 + \tfrac{56}{9}I_1 I_3^2 + \tfrac{56}{5}I_1^2 I_5 + \tfrac{256}{45}I_1^7\\
&\vdots \quad \vdots \quad \ddots
\end{aligned} \tag{3.81}$$

With the help of (3.78) we get that the integrals I_{2n+1} satisfy the following coupled nonlinear differential equations:

$$\begin{aligned}
&I_3'' + \frac{1}{x}I_3' - I_3 = 8I_1^3\\
&I_5'' + \frac{1}{x}I_5' - I_5 = \frac{40}{3}I_1^2 I_3 + \frac{32}{3}I_1^5\\
&I_7'' + \frac{1}{x}I_7' - I_7 = \frac{224}{9}I_1^4 I_3 + \frac{56}{9}I_1 I_3^2 + \frac{56}{5}I_1^2 I_5 + \frac{256}{45}I_1^7\\
&\qquad\qquad \vdots \quad \vdots \quad \ddots
\end{aligned} \tag{3.82}$$

So, the rhs of these equations is what makes the difference between $K_0(x)$ and the I_{2n+1} (see (3.73)).

We did not find an independent analytical verification of the relations (3.82). We have made a careful numerical check of them, and they are satisfied. It should be added that an analytical treatment of relations similar to (3.82) can be found in [6].

Therefore, assuming the analytical validity of (3.82), we conclude that the configuration (2.65) is indeed a solution of (1.1).

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Appendix. Hyperelliptic Riemann surfaces and delta function

Let us consider the equation for an hyperelliptic Riemann surface

$$y^2 = a(z) = (z - z_1)(z - z_2)\cdots(z - z_n). \tag{A.1}$$

There are branch points at $z = z_1, \ldots, z = z_n$. Let us suppose they are all distinct. Also the point at infinity is a branch point when n is odd. For simplicity let us suppose that n is even. There are two sheets, which are two copies of the z-plane. Now we draw a cut from z_1 to z_2, another from z_3 to z_4 and so on. The two sheets are now attached to each other through the cuts. By proceeding along a small circle around a branch point we will pass from one sheet to another and after 4π we are back to the initial point. Passing to the string interpretation, it is evident that now we have a string of length 4π instead of 2π as initially. The string interpretation is as follows: we have initially two strings that interact successively $n-2$ times and finally split into two separate strings as in the initial state. The Riemann surface we get in this way is an hyperelliptic one with two punctures representing the initial strings, two representing the final strings and $(n-2)/2$ handles. Let us call it Σ.

Let us return to (A.1) where y and z are coordinates of two complex planes, but, of course, they can be considered as functions over Σ. The coordinate z is not a good coordinate near a branch point. A good local coordinate near a branch point z_i is $\xi_i = \sqrt{z - z_i}$. That is, near z_i we have $z = z_i + {\xi_i}^2$. z is not a good coordinate at infinity either, it must be replaced by $w = 1/z$. After these substitutions we see that y is a meromorphic function, with n zeros at the branch points and a pole of order n at $z = \infty$ on each sheet.

Let us now consider the differential $\mathrm{d}z$. $\mathrm{d}z \sim \xi_i\, \mathrm{d}\xi_i$ near z_i, therefore, $\mathrm{d}z$ has simple zeros at the branch points. At infinity $\mathrm{d}z \sim w^{-2}\,\mathrm{d}w$, therefore, it has a double pole there, on both sheets.

Therefore, the product $y\,\mathrm{d}z$ is a meromorphic 1-form over Σ, with a single pole of order $n+2$ at infinity. It makes sense to integrate this form along a path, and this is what we do when we write $\zeta = \int \sqrt{a}\,\mathrm{d}z$, equation (2.47). As a consequence ζ is a function over Σ. It is worth giving a simple example. Let us consider the case in which $a = z - z_0$. In this case

$$\frac{\partial \zeta}{\partial z} = \frac{\sqrt{z - z_0}}{z} \tag{A.2}$$

and the approximate expression of ζ in terms of z near $z = z_0, 0, \infty$ is given by

$$\begin{aligned} &\zeta \sim \frac{2}{3z_0}(z - z_0)^{3/2} && \text{for} \quad z \sim z_0 \\ &\zeta \sim \frac{1}{2}\sqrt{z} && \text{for} \quad |z| \gg |z_0| \\ &\zeta \sim \sqrt{-z_0}\ln z && \text{for} \quad |z| \ll |z_0|. \end{aligned} \tag{A.3}$$

We see that problems may arise by taking ζ as a coordinate over Σ in correspondence with the branch points, as well as at $z = 0, \infty$. Since our calculations are mostly done by using ζ as a coordinate, we must be very careful when evaluations are involved near these points. In order to get the right results, we must always pass, at least tacitly, to good coordinates.

With this reservation in mind, let us come to equation (2.60)

$$u \sim -\frac{2}{\pi} f \ln|\zeta| \tag{A.4}$$

and to the behaviour

$$u = -\tfrac{1}{2}\ln|a| \tag{A.5}$$

which is required for consistency near a branch point. Near a generic branch point $a \sim z - z_0$, a good coordinate is $\xi = \sqrt{a}$. In terms of this coordinate we have

$$a \sim \xi^2 \qquad \zeta = \int \mathrm{d}z\sqrt{a} \sim \int \xi^2\,\mathrm{d}\xi \sim \xi^3 \qquad a \sim \zeta^{2/3}. \tag{A.6}$$

Therefore, in order that (A.4) be consistent with (A.5) we must have $f = \frac{\pi}{6}$.

Some attention must be paid to the definition of the delta functions, in order to avoid possible inconsistencies in fixing the value of f. Let us see this point in detail.

Consider the good coordinate ξ and

$$\int \mathrm{d}^2\xi\ \partial_\xi \partial_{\bar{\xi}} \ln|\xi| = \frac{\pi}{2} \int \mathrm{d}^2\xi\ \delta(\xi, \bar{\xi}). \tag{A.7}$$

Since the contribution to the integral is only at the origin we can restrict it to the unit disc around the origin, and proceed in another way by applying Stokes theorem

$$\int d^2\xi\, \partial_\xi \partial_{\bar\xi} \ln|\xi| = \frac{1}{2}\int d^2\xi\, \partial_{\bar\xi}\frac{1}{\xi} = \frac{1}{2}\oint d\xi \frac{1}{\xi} = \frac{1}{2}\int_0^{2\pi} d\theta = \pi i \tag{A.8}$$

where the contour integral extends over the unit circle around the origin and $\xi = e^{i\theta}$.

If we repeat the same calculation with the 'bad' coordinate a, we get

$$\int d^2a\, \partial_a \partial_{\bar a} \ln|a| = \frac{\pi}{2}\int d^2a\, \delta(a,\bar a) \tag{A.9}$$

and

$$\int d^2a\, \partial_a \partial_{\bar a} \ln|a| = \frac{1}{2}\int d^2a\, \partial_{\bar a}\frac{1}{a} = \frac{1}{2}\oint da \frac{1}{a} = \frac{4\pi i}{2} = 2\pi i. \tag{A.10}$$

The last steps are due to the fact that the angular integration for a extends over 4π since $a \sim \xi^2$.

In a similar way for ζ we will get

$$\int d^2\zeta\, \partial_\zeta \partial_{\bar\zeta} \ln|\zeta| = \frac{1}{2}\int d^2\zeta\, \partial_{\bar\zeta}\frac{1}{\zeta} = \frac{1}{2}\oint d\zeta \frac{1}{\zeta} = \frac{1}{2}\int_0^{6\pi} d\theta = 3\pi i. \tag{A.11}$$

At this point it is judicious to make use of different symbols for these delta functions: $\delta(\xi,\bar\xi)$, which is the usual delta function, and $\delta_a(a,\bar a)$, $\delta_\zeta(\zeta,\bar\zeta)$ so that, roughly speaking,

$$\delta_a(a,\bar a) \sim 2\delta(\xi,\bar\xi) \qquad \delta_\zeta(\zeta,\bar\zeta) \sim 3\delta(\xi,\bar\xi). \tag{A.12}$$

In addition, we must take into account the Jacobian factor due to the change of coordinates (a delta function transforms like the component of a 1-form). In conclusion we have the relation

$$\delta_\zeta(\zeta,\bar\zeta) = \tfrac{3}{2}\delta_a(a,\bar a)\partial_\zeta a \partial_{\bar\zeta}\bar a. \tag{A.13}$$

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